

Large deviation principle of SDEs with non-Lipschitzian coefficients under localized conditions

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Abstract

Localized sufficient conditions for the large deviation principle of the given stochastic differential equations will be presented for stochastic differential equations with non-Lipschitzian and time-inhomogeneous coefficients, which is weaker than those relevant conditions existing in the literature. We consider at first the large deviation principle when $\int_0^t \sup_{x \in \mathbb{R}^d} \|\sigma(s, x)\| \vee |b(s, x)| ds =: C_t < \infty$ for any fixed t , then we generalize the conclusion to unbounded case by using bounded approximation program.

MSC 2010: 60H10, 60F10.

Key words: stochastic differential equations; non-Lipschitzian; Euler approximation; large deviation principle; test function.

1 Introduction and Main Results

Let (Ω, \mathcal{F}, P) be a probability space, endowed with a complete filtration $(\mathcal{F}_t)_{t \geq 0}$. Consider the following stochastic differential equations (SDEs for short):

$$dX_t^\varepsilon = \varepsilon^{\frac{1}{2}} \sigma(t, X_t^\varepsilon) dB_t + b(t, X_t^\varepsilon) dt, \quad X_0^\varepsilon = x_0 \in \mathbb{R}^d \quad (1.1)$$

where ε is an arbitrary positive number, B_t is an m -dimensional standard \mathcal{F}_t -Brownian motion, and σ and b are \mathcal{F}_t -adapted functions from $\mathbb{R} \times \mathbb{R}^d$ to $\mathbb{R}^d \otimes \mathbb{R}^m$ and \mathbb{R}^d , respectively. Without loss of generality, we restrict ourselves that t is on the interval $[0, 1]$. Let μ_ε be the law of the solution of (1.1) on the space $C_{x_0}([0, 1], \mathbb{R}^d)$ of continuous functions starting from $x_0 \in \mathbb{R}^d$. When the coefficients are time-homogeneous, Fang and Zhang [11] got the large deviation principle under certain global non-Lipschitzian conditions. In [12], the second named author generalized Fang and Zhang's result to a more general time-inhomogeneous case. However, both of [11, 12] were concerned with global conditions. The aim of this paper

*Corresponding author. Email: langq@mail.buct.edu.cn. Supported by China Scholarship Council, National Natural Science Foundation of China (NSFC11026142) and Beijing Higher Education Young Elite Teacher Project (YETP0516).

is to get the large deviation principle under localized conditions which are even weaker than local Lipschitzian conditions (see the following (1.5)).

Much of the earlier work on the large deviation principle (see e.g. Freidlin and Wentzell [20], Donsker and Varadhan [8, 9]) was based on change of measure techniques, where a new measure is identified under which the events of interest have high probability, and then the probability of that event under the original probability measure is calculated using the Radon-Nikodym derivative. An approach analogous to the Prohorov compactness approach to weak convergence has been developed by Pukhalskii [17], O'Brien and Vervaat [15], de Acosta [1]. In [5], Chiarini and Fischer got the sufficient conditions of the so-called Laplace principle for stochastic differential equations when the coefficients also depend on ε and the past of solution trajectory. The proof is based on the weak convergence approach introduced by Dupuis and Ellis [10]. For more results about large deviation principle, one can see [2, 3, 4, 19], and references therein.

In order that the integrals in the definition of the solutions of the equation (1.1) are well-defined, we make the following assumption which is enforced throughout the paper

$$\int_0^T \sup_{|x| \leq R} (|b(s, x)| + \|\sigma(s, x)\|^2) ds < \infty, \quad \forall T, R > 0. \quad (1.2)$$

We also assume throughout the paper that the coefficients σ and b satisfy

$$\|\sigma(t, x) - \sigma(t, y)\| + |b(t, x) - b(t, y)| \leq G(t)H(|x - y|), \quad (1.3)$$

where $H : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an increasing continuous function with $H(0) = 0$, and $0 \leq G \in L^2([0, t])$ for any fixed $t > 0$. $|\cdot|$ denotes the Euclidean distance and $\langle \cdot, \cdot \rangle$ inner product in \mathbb{R}^d , $\|\sigma\|^2 = \sum_{i=1}^d \sum_{j=1}^m \sigma_{ij}^2$.

We consider the sufficient conditions of large deviation principle of stochastic differential equation (1.1). Assume that for any fixed $R > 0$, $\eta_R : [0, 1] \rightarrow \mathbb{R}_+$ is a differentiable function satisfying $\int_{0+} \frac{dx}{\eta_R(x)} = \infty$ with $\eta_R(0) = 0$, $\eta'_R \geq 0$, and f is nonnegative function such that $\int_0^t f^2(s) ds < \infty, \forall t > 0$.

Then we have the following result

Theorem 1.1 *Let $R > 0$ be fixed arbitrarily. Assume that*

$$\int_0^t \sup_{x \in \mathbb{R}^d} (|\sigma(s, x)| \vee |b(s, x)|) ds < \infty \quad (1.4)$$

for any fixed $t > 0$. If for any $|x| \vee |y| \leq R$, the following condition

$$\left(\|\sigma(t, x) - \sigma(t, y)\|^2 + 2\langle x - y, b(t, x) - b(t, y) \rangle \right) \vee \left| (\sigma(t, x) - \sigma(t, y))^T (x - y) \right| \leq f(t) \eta_R(|x - y|^2) \quad (1.5)$$

holds with $|x - y| < c_0 (< 1)$, $t \in [0, 1]$, (here and from now on A^T denotes the transpose of a matrix A), then the distribution family $\{\mu_\varepsilon, \varepsilon > 0\}$ satisfies a large deviation principle with the following good rate function

$$I(h) = \inf \left\{ \frac{1}{2} \int_0^1 |l'(t)|^2 dt : F(l) = h, l \in C_0([0, 1], \mathbb{R}^m) \right\}, \quad h \in C_{x_0}([0, 1], \mathbb{R}^d),$$

where l' denotes the gradient of l , and $F(l)$ satisfies the auxiliary ordinary differential equation

$$F(l)(t) = x_0 + \int_0^t b(s, F(l)(s))ds + \int_0^t \sigma(s, F(l)(s))l'(s)ds \quad (1.6)$$

for $l \in C_0([0, 1], R^m), t > 0$.

To prove this result, we need the following lemmas.

Suppose $X_n^\varepsilon(t)$ is Euler approximation of $X^\varepsilon(t)$ defined as

$$X_n^\varepsilon(t) := x_0 + \int_0^t b(s, X_n^\varepsilon(\frac{[ns]}{n}))ds + \varepsilon^{\frac{1}{2}} \int_0^t \sigma(s, X_n^\varepsilon(\frac{[ns]}{n}))dB_s \quad (1.7)$$

where the $[ns]$ denotes the integral part of ns .

Lemma 1.1 Assume that σ and b are bounded and that (1.5) holds. Then for any $\delta_0 > 0$, we have

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \varepsilon \log P(\sup_{0 \leq t \leq 1} |X^\varepsilon(t) - X_n^\varepsilon(t)| \geq \delta_0) = -\infty.$$

Let $l \in C_0([0, 1], R^m)$, define

$$e(l) := \begin{cases} \int_0^1 |l'(t)|^2 dt, & \text{if } l \text{ is absolutely continuous,} \\ +\infty, & \text{otherwise.} \end{cases}$$

and F_n be the Euler approximation of F , namely,

$$F_n(l)(t) := x_0 + \int_0^t b(s, F_n(l)(\frac{[ns]}{n}))ds + \int_0^t \sigma(s, F_n(l)(\frac{[ns]}{n}))l'(s)ds. \quad (1.8)$$

Lemma 1.2 Assume that σ and b are bounded and satisfy

$$\langle x - y, b(t, x) - b(t, y) \rangle \vee \left| (\sigma(t, x) - \sigma(t, y))^T (x - y) \right| \leq f(t) \eta_R(|x - y|^2) \quad (1.9)$$

where η_R and f are same as in condition (1.5). Then for any $\alpha > 0$,

$$\lim_{n \rightarrow \infty} \sup_{\{l: e(l) \leq \alpha\}} \sup_{0 \leq t \leq 1} |F_n(l)(t) - F(l)(t)| = 0.$$

Notice that condition (1.9) is weaker than (1.5).

From now on consider the general case that σ and b only satisfy the integrable condition (1.2). Let $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a differentiable function such that $\gamma'(x) \geq 0$, $\lim_{s \rightarrow \infty} \gamma(s) = \infty$, and $\int_K^\infty \frac{ds}{\gamma(s)+1} = \infty$ holds for some $K > 0$, g is a nonnegative function such that $\int_0^t g^2(s)ds < \infty$.

Lemma 1.3 If there exists $K > 0$ such that

$$\left(\|\sigma(t, x)\|^2 + 2\langle x, b(t, x) \rangle \right) \vee |\sigma^T(t, x)x| \leq g(t)(\gamma(|x|^2) + 1) \quad (1.10)$$

holds for any $|x| \geq K$, $t \in [0, 1]$, then

$$\lim_{R \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \varepsilon \log P(\sup_{0 \leq t \leq 1} |X^\varepsilon(t)| \geq R) = -\infty.$$

Lemma 1.4 *Let $\alpha > 0$. Assume that (1.10) holds. Then*

$$\sup_{\{l: e(l) \leq \alpha\}} \sup_{0 \leq t \leq 1} |F(l)(t)| < \infty.$$

Note that, by the results we have proved in [13, 14], if σ and b satisfy conditions (1.5) and (1.10), then for any fix $\varepsilon > 0$, stochastic differential equation (1.1) has a unique global strong solution.

Lemma 1.5 *$I(\cdot)$ defined as in Theorem 1.1 is a good rate function on $C_{x_0}([0, 1], R^d)$, that is, for any $\alpha > 0$, the sublevel set $\{h \in C_{x_0}([0, 1], R^d) : I(h) \leq \alpha\}$ is compact.*

Theorem 1.2 *Suppose σ and b satisfy conditions (1.5) and (1.10). Then the distribution family $\{\mu_\varepsilon, \varepsilon > 0\}$ of solutions of SDEs (1.1) satisfy a large deviation principle with the same good rate function as in Theorem 1.1.*

Remark 1.1 *In [5], the authors got the large deviation principle for the Itô SDE under assumptions H1-H6, but the uniformly continuous condition H1 does not hold in our case since $G(t)$ is only locally integrable but not bounded in t , and tightness condition H6 does not hold either since they used the sublinear growth condition of σ and b to prove the tightness condition (see (1.5) and (1.10)). So their Theorem 2.1 can not be used under our condition. Moreover, their assumptions are not easy to check. On the other hand, we can take $\eta_R(x) = Rx \log \frac{1}{x}$, $x \leq \frac{1}{e}$ and $\gamma(x) = x \log x$, $x \geq K(> 1)$ for some K large enough, σ and b may not satisfy the Lipschitz condition, so the method of moments used in the literature (such as [6, 7, 16]) does not work here because of the non-Lipschitz feature of coefficients.*

We now give an example to show that our conditions are really weaker than those relevant conditions existing in the literature.

Example 1.1 *We just consider the time-homogeneous case for simplicity. Suppose $d = 2$, $m = 1$. For any $r > 0$, define*

$$\sigma(x) = |x|^r (-x_2, x_1)^T, b(x) = -|x|^{2r} x^T.$$

Since the local Lipschitzian condition holds for both σ and b , the condition (1.5) holds naturally. It's obvious that there exists a unique strong solution for the giving stochastic differential equation. On the other hand,

$$\left(|\sigma(x)|^2 + 2\langle x, b(x) \rangle \right) \vee |\sigma^T(x)x| = (|x|^{2r+2} - 2|x|^{2r+2}) \vee 0 = 0 \leq K(|x|^2 + 1).$$

So by Theorem 1.2, we know that the large deviation principle holds in this case. But there is NO constant $C > 0$ such that

$$|\sigma(x)|^2 = |x|^{2r+2} \leq C(|x|^2 \log |x| + 1)$$

holds for $|x|$ large enough. So we have given a sufficient condition for large deviation principle which is weaker than that of [11].

The rest of the paper is organized as follows. For the case that $\sigma(t, x)$ and $b(t, x)$ satisfy condition (1.4), we will first show Lemma 1.1 in Section 2. Then we prove that the Euler approximation (1.8) converges uniformly to the solution of auxiliary equation (1.6) in Section 3. In Section 4, we will drop the assumption (1.4) and get the large deviation principle by bounded approximation in general case.

2 Proof of Lemma 1.1

Let $X_n^\varepsilon(t)$ be the Euler approximation of $X^\varepsilon(t)$ defined as (1.7). Denote

$$Y_n^\varepsilon(t) := X_n^\varepsilon(t) - X^\varepsilon(t), \quad \xi_n^\varepsilon(t) := |Y_n^\varepsilon(t)|^2.$$

Define the following test function

$$\varphi_{\rho,\lambda}(x) := \exp\left(\lambda \int_0^x \frac{ds}{\eta_R(s) + \rho}\right).$$

Then

$$\varphi'_{\rho,\lambda}(x) = \frac{\lambda \varphi_{\rho,\lambda}(x)}{\eta_R(x) + \rho},$$

and

$$\varphi''_{\rho,\lambda}(x) = \frac{\lambda^2 \varphi_{\rho,\lambda}(x) - \lambda \varphi_{\rho,\lambda}(x) \eta'_R(x)}{(\eta_R(x) + \rho)^2} \leq \frac{\lambda^2 \varphi_{\rho,\lambda}(x)}{(\eta_R(x) + \rho)^2}.$$

We have used the condition $\eta'_R \geq 0$ in the above inequality. Introduce stopping times

$$\tau_n^\varepsilon := \inf\{t > 0, |X_n^\varepsilon(t) - X_n^\varepsilon(\frac{[nt]}{n})| \geq \delta\}$$

where $\delta > 0$ is an arbitrarily small number,

$$\tau_R := \inf\{t > 0, |X_n^\varepsilon(t)| \vee |X^\varepsilon(t)| \geq R\}$$

and

$$T_n^\varepsilon := \inf\{t > 0, |\xi_n^\varepsilon(t)| \geq \delta_0^2\}.$$

Since σ and b satisfy integrable condition (1.4), it's obvious that $\lim_{R \rightarrow \infty} \tau_R = \infty$.

Without loss of generality, suppose $0 < \delta_0 \leq c_0$, then it follows by Itô's formula that

$$\begin{aligned} \varphi_{\rho,\lambda}(\xi_n^\varepsilon(t)) &= 1 + 2\varepsilon^{\frac{1}{2}} \int_0^t \varphi'_{\rho,\lambda}(\xi_n^\varepsilon(s)) \langle Y_n^\varepsilon(s), (\sigma(s, X^\varepsilon(s)) - \sigma(s, X_n^\varepsilon(\frac{[ns]}{n}))) dB_s \rangle \\ &\quad + 2 \int_0^t \varphi'_{\rho,\lambda}(\xi_n^\varepsilon(s)) \langle Y_n^\varepsilon(s), b(s, X^\varepsilon(s)) - b(s, X_n^\varepsilon(\frac{[ns]}{n})) \rangle ds \\ &\quad + \varepsilon \int_0^t \varphi'_{\rho,\lambda}(\xi_n^\varepsilon(s)) \|\sigma(s, X^\varepsilon(s)) - \sigma(s, X_n^\varepsilon(\frac{[ns]}{n}))\|^2 ds \\ &\quad + 2\varepsilon \int_0^t \varphi''_{\rho,\lambda}(\xi_n^\varepsilon(s)) |(\sigma(s, X^\varepsilon(s)) - \sigma(s, X_n^\varepsilon(\frac{[ns]}{n})))^T Y_n^\varepsilon(s)|^2 ds. \end{aligned}$$

By definition of the stopping time, when $s \leq \tau_n^\varepsilon \wedge T_n^\varepsilon \wedge \tau_R$, it follows that $|Y_n^\varepsilon(s)| \leq \delta_0$ and $|X_n^\varepsilon(s) - X_n^\varepsilon(\frac{[ns]}{n})| \leq \delta$. So

$$\begin{aligned} &\langle Y_n^\varepsilon(s), b(s, X^\varepsilon(s)) - b(s, X_n^\varepsilon(\frac{[ns]}{n})) \rangle \\ &= \langle Y_n^\varepsilon(s), b(s, X^\varepsilon(s)) - b(s, X_n^\varepsilon(s)) \rangle + \langle Y_n^\varepsilon(s), b(s, X_n^\varepsilon(s)) - b(s, X_n^\varepsilon(\frac{[ns]}{n})) \rangle \\ &\leq \langle Y_n^\varepsilon(s), b(s, X^\varepsilon(s)) - b(s, X_n^\varepsilon(s)) \rangle + \delta_0 G(s) H(\delta). \end{aligned} \tag{2.1}$$

Similarly,

$$\begin{aligned}
& \|\sigma(s, X^\varepsilon(s)) - \sigma(s, X_n^\varepsilon(\frac{[ns]}{n}))\|^2 \\
& \leq 2\|\sigma(s, X^\varepsilon(s)) - \sigma(s, X_n^\varepsilon(s))\|^2 + 2\|\sigma(s, X_n^\varepsilon(s)) - \sigma(s, X_n^\varepsilon(\frac{[ns]}{n}))\|^2 \\
& \leq 2\|\sigma(s, X^\varepsilon(s)) - \sigma(s, X_n^\varepsilon(s))\|^2 + 2G^2(s)H^2(\delta).
\end{aligned}$$

When $\varepsilon \leq \frac{1}{2}$, by (1.5) we arrive at

$$\begin{aligned}
& 2\langle Y_n^\varepsilon(s), b(s, X^\varepsilon(s)) - b(s, X_n^\varepsilon(\frac{[ns]}{n})) \rangle + \varepsilon\|\sigma(s, X^\varepsilon(s)) - \sigma(s, X_n^\varepsilon(\frac{[ns]}{n}))\|^2 \\
& \leq 2\langle Y_n^\varepsilon(s), b(s, X^\varepsilon(s)) - b(s, X_n^\varepsilon(s)) \rangle + 2\varepsilon\|\sigma(s, X^\varepsilon(s)) - \sigma(s, X_n^\varepsilon(s))\|^2 \\
& \quad + 2G(s)H(\delta)\delta_0 + 2\varepsilon G^2(s)H^2(\delta) \\
& \leq f(s)\eta_R(\xi_n^\varepsilon(s)) + 2G(s)H(\delta)\delta_0 + 2\varepsilon G^2(s)H^2(\delta) \\
& \leq (f(s) + 2(G(s) + 1)^2)[\eta_R(\xi_n^\varepsilon(s)) + H(\delta)(\delta_0 + \varepsilon H(\delta))].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left| \left[\sigma(s, X^\varepsilon(s)) - \sigma(s, X_n^\varepsilon(\frac{[ns]}{n})) \right]^T Y_n^\varepsilon(s) \right|^2 & \leq 2 \left| (\sigma(s, X^\varepsilon(s)) - \sigma(s, X_n^\varepsilon(s)))^T Y_n^\varepsilon(s) \right|^2 \\
& \quad + 2 \left| (\sigma(s, X_n^\varepsilon(s)) - \sigma(s, X_n^\varepsilon(\frac{[ns]}{n})))^T Y_n^\varepsilon(s) \right|^2 \\
& \leq 2f^2(s)\eta_R^2(\xi_n^\varepsilon(s)) + 2\delta_0^2 G^2(s)H^2(\delta) \\
& \leq 2(f^2(s) + G^2(s))[\eta_R(\xi_n^\varepsilon(s)) + \delta_0 H(\delta)]^2.
\end{aligned}$$

Take

$$\rho = H(\delta)(\delta_0 + \varepsilon H(\delta)).$$

Then $\rho \rightarrow 0$ as $\delta \rightarrow 0$. By the definition of $\varphi_{\rho, \lambda}$, we have

$$\begin{aligned}
\mathbb{E}\varphi_{\rho, \lambda}(\xi_n^\varepsilon(t \wedge \tau_n^\varepsilon \wedge T_n^\varepsilon \wedge \tau_R)) & \leq 1 + \lambda \mathbb{E} \int_0^{t \wedge \tau_n^\varepsilon \wedge T_n^\varepsilon \wedge \tau_R} (f(s) + 2(G(s) + 1)^2) \varphi_{\rho, \lambda}(\xi_n^\varepsilon(s)) ds \\
& \quad + 4\varepsilon \lambda^2 \mathbb{E} \int_0^{t \wedge \tau_n^\varepsilon \wedge T_n^\varepsilon \wedge \tau_R} (f^2(s) + G^2(s)) \varphi_{\rho, \lambda}(\xi_n^\varepsilon(s)) ds.
\end{aligned}$$

By Gronwall's lemma, it follows that (letting $R \rightarrow \infty$ and taking $t = 1$),

$$\mathbb{E}\varphi_{\rho, \lambda}(\xi_n^\varepsilon(1 \wedge \tau_n^\varepsilon \wedge T_n^\varepsilon)) \leq \exp\left\{\lambda \int_0^1 (f + 2(G + 1)^2)(s) ds + 4\varepsilon \lambda^2 \int_0^1 (f^2 + G^2)(s) ds\right\}.$$

On the other hand,

$$\begin{aligned}
\mathbb{E}\varphi_{\rho, \lambda}(\xi_n^\varepsilon(1 \wedge \tau_n^\varepsilon \wedge T_n^\varepsilon)) & \geq \mathbb{E}(\varphi_{\rho, \lambda}(\xi_n^\varepsilon(T_n^\varepsilon))I_{\{\tau_n^\varepsilon \geq 1, T_n^\varepsilon \leq 1\}}) \\
& = \varphi_{\rho, \lambda}(\delta_0^2)P(\tau_n^\varepsilon \geq 1, T_n^\varepsilon \leq 1).
\end{aligned}$$

Taking $\lambda = \frac{1}{\varepsilon}$. It follows that

$$\varepsilon \log P(\tau_n^\varepsilon \geq 1, T_n^\varepsilon \leq 1) \leq C - \int_0^{\delta_0^2} \frac{ds}{\eta(s) + \rho},$$

where $C := \int_0^1 (f(s) + 4f^2(s))ds + \int_0^1 (6G^2(s) + 4G(s) + 2)ds$. Therefore

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log P(\tau_n^\varepsilon \geq 1, T_n^\varepsilon \leq 1) \leq C - \int_0^{\delta_0^2} \frac{ds}{\eta(s) + \rho}.$$

Now

$$P(\sup_{0 \leq t \leq 1} |X_t^\varepsilon - X_n^\varepsilon(t)| \geq \delta_0) = P(T_n^\varepsilon \leq 1) \leq P(\tau_n^\varepsilon \geq 1, T_n^\varepsilon \leq 1) + P(\tau_n^\varepsilon \leq 1).$$

Since σ and b satisfy integrable condition (1.4), then

$$\begin{aligned} P(\tau_n^\varepsilon \leq 1) &\leq \sum_{k=1}^n P(\sup_{\frac{k-1}{n} \leq t < \frac{k}{n}} |X_n^\varepsilon(t) - X_n^\varepsilon(\frac{k-1}{n})| \geq \delta) \\ &\leq \sum_{k=1}^n P(\sup_{\frac{k-1}{n} \leq t < \frac{k}{n}} \varepsilon^{\frac{1}{2}} |\int_{\frac{k-1}{n}}^t \sigma(s, X_n^\varepsilon(\frac{k-1}{n})) dB_s| \geq \delta - B_{n,k}), \end{aligned}$$

where $B_{n,k} := \int_{\frac{k-1}{n}}^{\frac{k}{n}} \sup_{x \in \mathbb{R}} |b(t, x)| dt$, $B := \sum_{k=1}^n B_{n,k}$. Similarly, define $A_{n,k}$ and A with $|b|$ replaced by $\|\sigma\|^2$. By (1.4) again, it's obvious that

$$\lim_{n \rightarrow \infty} \sup_{k \leq n} (A_{n,k} \vee B_{n,k}) = 0. \quad (2.2)$$

Since

$$M_t := \varepsilon^{\frac{1}{2}} \int_{\frac{k-1}{n}}^{t + \frac{k-1}{n}} \sigma(s, X_n^\varepsilon(\frac{k-1}{n})) dB_s$$

is a martingale with respect to $\tilde{\mathcal{F}}_t := \mathcal{F}_{t + \frac{k-1}{n}}$ for $0 \leq t < \frac{1}{n}$, then by exponential martingale inequality, for any d dimensional vector $|\theta| = 1$, we have

$$\begin{aligned} P(\sup_{0 \leq t < \frac{1}{n}} \langle \theta, M_t \rangle \geq \delta - B_{n,k}) &\leq P(\sup_{0 \leq t < \frac{1}{n}} \alpha \langle \theta, M_t \rangle - \frac{\alpha^2}{2} \langle \langle \theta, M \rangle \rangle_t \geq \alpha(\delta - B_{n,k}) - \frac{\varepsilon \alpha^2 A_{n,k}}{2}) \\ &\leq \exp(-\alpha(\delta - B_{n,k}) + \frac{\varepsilon \alpha^2 A_{n,k}}{2}), \end{aligned}$$

where $\langle \langle \theta, M \rangle \rangle_t$ denotes the quadratic variation process of $\langle \theta, M_t \rangle$. Taking $\alpha = \frac{\delta - B_{n,k}}{\varepsilon A_{n,k}}$,

$$P(\sup_{0 \leq t < \frac{1}{n}} \langle \theta, M_t \rangle \geq \delta - B_{n,k}) \leq \exp(-\frac{\delta - B_{n,k}}{2\varepsilon A_{n,k}}) \leq \exp(-\frac{\delta - \sup_{k \leq n} B_{n,k}}{2\varepsilon \sup_{k \leq n} A_{n,k}}).$$

We have used the fact (2.2) here. Then by Stroock [18], we have

$$P(\sup_{0 \leq t < \frac{1}{n}} \varepsilon^{\frac{1}{2}} |\int_{\frac{k-1}{n}}^t \sigma(s, X_n^\varepsilon(\frac{k-1}{n})) dB_s| \geq \delta - B_{n,k}) \leq 2d \exp(-\frac{\delta - \sqrt{d} \sup_{k \leq n} B_{n,k}}{2\varepsilon d \sup_{k \leq n} A_{n,k}}).$$

Thus,

$$P(\tau_n^\varepsilon \leq 1) \leq 2nd \exp(-\frac{\delta - \sqrt{d} \sup_{k \leq n} B_{n,k}}{2\varepsilon d \sup_{k \leq n} A_{n,k}}).$$

For sufficiently large n , it follow that

$$\varepsilon \log P(\tau_n^\varepsilon \leq 1) \leq \varepsilon \log(2nd) - \frac{\delta - \sqrt{d} \sup_{k \leq n} B_{n,k}}{2d \sup_{k \leq n} A_{n,k}} \leq -\frac{\delta}{4d \sup_{k \leq n} A_{n,k}}.$$

Since n is independent of ε , then

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \varepsilon \log P\left(\sup_{0 \leq t \leq 1} |X_t^\varepsilon - X_n^\varepsilon(t)| \geq \delta_0\right) \\ & \leq \limsup_{\varepsilon \rightarrow 0} \varepsilon \log(P(\tau_n^\varepsilon \geq 1, T_n^\varepsilon \leq 1) + P(\tau_n^\varepsilon \leq 1)) \\ & \leq \limsup_{\varepsilon \rightarrow 0} \varepsilon \log P(\tau_n^\varepsilon \geq 1, T_n^\varepsilon \leq 1) \vee \limsup_{\varepsilon \rightarrow 0} \varepsilon \log P(\tau_n^\varepsilon \leq 1) \\ & \leq (C - \int_0^{\delta_0^2} \frac{ds}{\eta(s) + \rho}) \vee (-\frac{\delta}{4d \sup_{k \leq n} A_{n,k}}). \end{aligned}$$

By letting $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \varepsilon \log P\left(\sup_{0 \leq t \leq 1} |X_t^\varepsilon - X_n^\varepsilon(t)| \geq \delta_0\right) \leq C - \int_0^{\delta_0^2} \frac{ds}{\eta(s) + \rho}.$$

Since $\rho \rightarrow 0$ as $\delta \rightarrow 0$, taking limit with δ , the right hand side of the inequality tends to $-\infty$. We complete the proof. \square

3 Proof of Lemma 1.2

Define the test function

$$\varphi_\rho(x) := \exp\left(\int_0^x \frac{ds}{\eta_R(s) + \rho}\right).$$

Let

$$Y_n^l(t) := F_n(l)(t) - F(l)(t), \quad Z_n^l(t) := |Y_n^l(t)|^2.$$

For any l with $e(l) \leq \alpha$ and $\delta > 0$ small enough (less than 1), define

$$\tau_n(l) := \inf\{t \geq 0, |Y_n^l(t)| > \delta\}$$

and

$$\tau_R := \inf\{t, |F_n(l)(t)| \vee |F(l)(t)| > R\}.$$

As in the proof of Lemma 1.1, by (1.4), it's clear that $\tau_R \rightarrow \infty$ as $R \rightarrow \infty$.

Denote

$$\begin{aligned} e_s &:= b(s, F_n(l)(\frac{[ns]}{n})) - b(s, F(l)(s)) \\ h_s &:= \sigma(s, F_n(l)(\frac{[ns]}{n})) - \sigma(s, F(l)(s)) \end{aligned}$$

Then by the chain rule, we have

$$\begin{aligned}\varphi_\rho(Z_n^l(t \wedge \tau_n(l) \wedge \tau_R)) &= 1 + 2 \int_0^{t \wedge \tau_n(l) \wedge \tau_R} \varphi'_\rho(Z_n^l(s)) \langle Y_n^l(s), e_s \rangle ds \\ &\quad + 2 \int_0^{t \wedge \tau_n(l) \wedge \tau_R} \varphi'_\rho(Z_n^l(s)) \langle Y_n^l(s), h_s l'(s) \rangle ds.\end{aligned}$$

Since σ, b are bounded with x for any fixed t , it follows that

$$\begin{aligned}|F_n(l)(t) - F_n(l)(\frac{[nt]}{n})| &\leq \int_{\frac{[nt]}{n}}^t |b(s, F_n(l)(\frac{[ns]}{n}))| ds + \int_{\frac{[nt]}{n}}^t \|\sigma(s, F_n(l)(\frac{[ns]}{n}))\| |l'(s)| ds \\ &\leq C_\alpha \frac{1}{\sqrt{n}},\end{aligned}$$

where $0 < C_\alpha$ is independent of n . We have used Hölder inequality in the last step. As in proof of Lemma 1.1, for $s \leq t \wedge \tau_n(l) \wedge \tau_R$, we have

$$\begin{aligned}\langle Y_n^l(s), e_s \rangle &= \langle Y_n^l(s), b(s, F_n(l)(\frac{[ns]}{n})) - b(s, F_n(l)(s)) \rangle \\ &\quad + \langle Y_n^l(s), b(s, F_n(l)(s)) - b(s, F(l)(s)) \rangle \\ &\leq \delta G(s) H(\frac{C_\alpha}{\sqrt{n}}) + f(s) \eta(Z_n^l(s)) \\ &\leq (f(s) + G(s)) (\eta(Z_n^l(s)) + \delta H(\frac{C_\alpha}{\sqrt{n}})).\end{aligned}$$

Similarly, we have

$$\langle Y_n^l(s), h_s \rangle \leq (f(s) + G(s)) (\eta(Z_n^l(s)) + \delta H(\frac{C_\alpha}{\sqrt{n}})).$$

Take $\rho_n = \delta H(\frac{C_\alpha}{\sqrt{n}})$. It follows that

$$\varphi_{\rho_n}(Z_n^l(t \wedge \tau_n(l) \wedge \tau_R)) \leq 1 + 2 \int_0^{t \wedge \tau_n(l) \wedge \tau_R} \varphi_{\rho_n}(Z_n^l(s)) (f(s) + G(s)) (1 + |l'(s)|) ds.$$

Since

$$\int_0^t (f(s) + G(s)) |l'(s)| ds \leq \left(\int_0^t (f(s) + G(s))^2 ds \right)^{\frac{1}{2}} \left(\int_0^t |l'(s)|^2 ds \right)^{\frac{1}{2}} < \infty,$$

by Gronwall's lemma and letting $R \rightarrow \infty$, we have

$$\varphi_{\rho_n}(Z_n^l(1 \wedge \tau_n(l))) \leq \exp\{2 \int_0^1 (f(s) + G(s)) (1 + |l'(s)|) ds\}.$$

Take supremum with $l \in \{l : e(l) \leq \alpha\}$ and let $n \rightarrow \infty$, since $\varphi_\rho(x)$ is increasing in x , it follows that

$$\limsup_{n \rightarrow \infty} \varphi_{\rho_n}(\sup_{l: e(l) \leq \alpha} Z_n^l(1 \wedge \tau_n(l))) \leq \exp\{2 \int_0^1 (f(s) + G(s)) ds + 2\sqrt{2\alpha} \left(\int_0^1 (f(s) + G(s))^2 ds \right)^{\frac{1}{2}}\}.$$

Since $\{\tau_n(l) > 1\} = \{\sup_{0 \leq t \leq 1} |F_n(l)(t) - F(l)(t)| \leq \delta\}$, we only need to show $\tau_n(l) > 1$ for all $l \in \{l : e(l) \leq \alpha\}$ and n large enough. If not, there exists $\delta > 0$, a subsequence $\{n_k, k \geq 1\}$ of positive integers and $l_{n_k} \in \{l : e(l) \leq \alpha\}$ such that $\tau_{n_k}(l_{n_k}) \leq 1$. Then

$$\begin{aligned} \varphi_{\rho_{n_k}}(\delta^2) &= \varphi_{\rho_{n_k}}(Z_{n_k}^{l_{n_k}}(1 \wedge \tau_{n_k}(l_{n_k}))) \\ &\leq \exp\left\{2 \int_0^1 (f(s) + G(s))ds + 2\sqrt{2\alpha} \left(\int_0^1 (f(s) + G(s))^2 ds \right)^{\frac{1}{2}}\right\} < \infty. \end{aligned}$$

Let $k \rightarrow \infty$. Then by the definition of φ_ρ , the left hand side tends to ∞ . This is a contradiction. So we have

$$\sup_{l: e(l) \leq \alpha} \sup_{0 \leq t \leq 1} |F_n(l)(t) - F(l)(t)| \leq \delta$$

for any $\delta > 0$ for n large enough. We complete the proof. \square

Proof of Theorem 1.1 Let F_n and X_n^ε are Euler approximation of F and X^ε respectively with the same scale. Notice that $X_n^\varepsilon(s) = F_n(\sqrt{\varepsilon}B)(s)$, where B is the Brownian motion. It's clear that F_n is continuous for each n . Since it's well known that $\sqrt{\varepsilon}B$ satisfies large deviation principle, now according to Lemma 1.1 and Lemma 1.2 and Theorem 4.2.23 in [6], we know that our process X^ε also satisfies large deviation principle. We complete the proof. \square

4 Large deviation principle in the general case

Proof of Lemma 1.3 Define $\xi^\varepsilon(t) := |X^\varepsilon(t)|^2$, and

$$\varphi(x) := \exp\left(\lambda \int_0^x \frac{ds}{\gamma(s) + 1}\right).$$

Define $\tau_R^\varepsilon := \inf\{t > 0, \xi^\varepsilon(t) \geq R^2\}$, it's clear that $\tau_R \rightarrow \infty$ as $R \rightarrow \infty$ since the solution is non explosive under condition (1.10). Then by Itô's formula and taking expectation on both sides, it follows that

$$\begin{aligned} \mathbb{E}\varphi(\xi^\varepsilon(t \wedge \tau_R^\varepsilon)) &\leq 1 + \mathbb{E} \int_0^{t \wedge \tau_R^\varepsilon} \varphi'(\xi^\varepsilon(s))g(s)(\gamma(\xi^\varepsilon(s)) + 1)ds \\ &\quad + 2\varepsilon \mathbb{E} \int_0^{t \wedge \tau_R^\varepsilon} \varphi''(\xi^\varepsilon(s))g(s)(\gamma(\xi^\varepsilon(s)) + 1)^2 ds. \end{aligned}$$

By the definition of φ and Gronwall's lemma, it follows that

$$\mathbb{E}\varphi(\xi^\varepsilon(t \wedge \tau_R^\varepsilon)) \leq e^{T \int_0^t g(s)ds} \varphi(|x_0|^2)$$

where $T = 2\lambda^2\varepsilon + \lambda$. Let $t = 1$. Then

$$P(\tau_R^\varepsilon \leq 1)\varphi(R^2) \leq e^{T \int_0^1 g(s)ds} \varphi(|x_0|^2).$$

That is,

$$\varepsilon \log P(\tau_R^\varepsilon \leq 1) \leq T\varepsilon \int_0^1 g(s)ds - \lambda\varepsilon \int_{|x_0|^2}^{R^2} \frac{dx}{\gamma(x) + 1}.$$

Take $\lambda = \frac{1}{\varepsilon}$, and let $\varepsilon \rightarrow 0$ and $R \rightarrow \infty$ subsequently. We complete the proof. \square

Proof of Lemma 1.4 Define $Z^l(t) := |F(l)(t)|^2$, and

$$\varphi(x) := \exp\left(\int_0^x \frac{ds}{\gamma(s) + 1}\right).$$

It follows by (1.10) that,

$$\varphi(Z^l(t)) \leq \varphi(|x_0|^2) + \int_0^t g(s)(1 + 2|l'(s)|)\varphi(Z^l(s))ds.$$

Now by Gronwall's lemma, it follows that

$$\begin{aligned} \varphi(Z^l(t)) &\leq \varphi(|x_0|^2) \exp\left\{\int_0^t g(s)(1 + 2|l'(s)|)ds\right\} \\ &\leq \varphi(|x_0|^2) \exp\{(1 + 2\sqrt{\alpha})\sqrt{\int_0^1 g^2(s)ds}\} \end{aligned}$$

holds for any $l \in \{l, e(l) \leq \alpha\}$ and $0 \leq t \leq 1$. We have used Hölder inequality in the last inequality. Since φ is increasing, it follows that

$$\varphi\left(\sup_{e(l) \leq \alpha} \sup_{0 \leq t \leq 1} Z^l(t)\right) \leq \varphi(|x_0|^2) \exp\{(1 + 2\sqrt{\alpha})\sqrt{\int_0^1 g^2(s)ds}\}.$$

By the definition of φ , it follows that $\sup_{e(l) \leq \alpha} \sup_{0 \leq t \leq 1} Z^l(t) < \infty$. \square

In what follows, we will consider large deviation principle of solutions of stochastic differential equation (1.1) without the integrable condition (1.4) on σ and b . To this end, we will use the bounded approximation method.

For $R > 0$, define

$$m_R(t) = \sup_{|x| \leq R} \{|b(t, x)| \vee \|\sigma(t, x)\|\}.$$

Then $m_R \in L^2([0, t])$ for any $t > 0$ by (1.2). Let

$$b_i^R(t, x) := (-m_R(t) - 1) \vee b_i(t, x) \wedge (m_R(t) + 1),$$

$$\sigma_{ij}^R(t, x) := (-m_R(t) - 1) \vee \sigma_{ij}(t, x) \wedge (m_R(t) + 1)$$

and

$$b_R(t, x) := (b_1^R(t, x), \dots, b_d^R(t, x)), \quad \sigma_R(t, x) := (\sigma_{ij}^R(t, x)).$$

Then

$$b_R(t, x) = b(t, x), \quad \sigma_R(t, x) = \sigma(t, x), \quad \forall |x| \leq R, \quad t \in [0, 1].$$

It's obvious that b_R, σ_R satisfy (1.4), and satisfy (1.5) with the same f and η_R . Let X_R^ε be the solution of

$$dX_R^\varepsilon(t) = \varepsilon^{\frac{1}{2}} \sigma_R(t, X_R^\varepsilon(t)) dB_t + b_R(t, X_R^\varepsilon(t)) dt, \quad X_R^\varepsilon(0) = x_0.$$

For $l \in C_0([0, 1], R^m)$ with $e(l) < \infty$, let $F_R(l)(t)$ be the solution of

$$dF_R(l)(t) = \sigma_R(t, F_R(l)(t)) l'(t) dt + b_R(t, F_R(l)(t)) dt, \quad F_R(l)(0) = x_0. \quad (4.1)$$

If $\sup_{0 \leq t \leq 1} |F(l)(t)| \leq R$, where $F(l)$ is the solution of differential equation (1.6), then $F(l)$ is the solution of differential equation (4.1). By uniqueness of solutions, one can see that $F_R(l)(t) = F(l)(t)$ for $0 \leq t \leq 1$. Define

$$I_R(f) = \inf \left\{ \frac{1}{2} e(l) : F_R(l) = f \right\}, \quad f \in C_{x_0}([0, 1], R^d).$$

Then

$$I_R(f) = I(f), \quad f \in C_{x_0}([0, 1], R^d), \quad \sup_{0 \leq t \leq 1} |f(t)| \leq R.$$

By Theorem 1.1, $\{\mu_\varepsilon^R, \varepsilon > 0\}$ satisfies a large deviation principle (with $I_R(\cdot)$).

Proof of Lemma 1.5 By Lemma 1.4, for $\alpha > 0$, there exists $R > 0$ such that

$$\sup_{l: e(l) \leq \alpha} \sup_{0 \leq t \leq 1} |F(l)(t)| \leq R. \quad (4.2)$$

Thus

$$F_R(l) = F(l), \quad \forall l \in \{l : e(l) \leq \alpha\}.$$

Let $\{f_n\}$ be a sequence in $\{f : I(f) \leq \alpha\}$. Then there exists a sequence $l_n \in C_0([0, 1], R^m)$ such that $F(l_n) = f_n$ and $\frac{1}{2} e(l_n) \leq \alpha + \frac{1}{n}$. So there exists a limit point $l \in C_0([0, 1], R^m)$ of $\{l_n\}$ such that $\frac{1}{2} e(l) \leq \alpha$. According to (4.2) we have $F_R(l_n) = F(l_n) = f_n$, and $F_R(l_n)$ converges uniformly (over $[0, 1]$) to $F_R(l) = F(l)$ (up to a subsequence). Let $f = \lim_{n \rightarrow \infty} f_n = F(l)$. Then $I(f) = \frac{1}{2} e(l) \leq \alpha$. So $\{f : I(f) \leq \alpha\}$ is compact. \square

Proof of Theorem 1.2 Repeat the proof of theorem *E* in Fang and Zhang [11] word by word, we can prove that Theorem 1.2 holds under conditions (1.5) and (1.10), so we omit it here. \square

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